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PORTFOLIO OPTIMIZATION USING MEASURES OF CROSS ENTROPY

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ABSTRACT

It is a well-known fact that in the literature of information theory, a variety of divergence (distance or cross entropy) measures is available, each with its own merits and limitations. These measures are applicable to various disciplines of Mathematical Sciences. One such discipline pertaining to Operations Research is portfolio analysis. In the present communication, we have developed two new parametric measures of cross entropy and consequently provided the applications of these measures for the study of optimization principles for the development of measures of risk in portfolio analysis. We have observed that minimizing these measures implies the minimization of the expected utility of the risk-prone person and maximization of the expected utility of a risk-averse person.

Keywords: Portfolio selection theory, cross entropy, mean-variance efficient frontier, uncertainty.

INTRODUCTION

Markowitz (1952) introduced the modern portfolio selection theory, which deals with the relevant beliefs about future performances and ends with the choice of portfolio. We consider a fundamental rule that the investor should consider expected return a desirable thing and variance of return an undesirable thing. Markowitz (1952) illustrated geometrically relations between beliefs and choice of portfolio according to the "expected returns-variance of returns" rule. It is worth mentioning that some of the investments made by the investor may yield low returns, but these may be compensated by considerations of relative safety because of a proven record of non-volatility in price fluctuations. On the other hand, there might be some better investments which would be promising and achieve high expected returns, but these may be prone to a great deal of risk. However, investor's major problem is to find a satisfactory measure of risk. The earliest measure proposed for the return on all investments was variance and its proposal was based upon the fundamental argument that risk increases with variance. Accordingly, Markowitz (1952) introduced the concept of mean-variance efficient frontier, which enabled him to find all the possible efficient portfolios that simultaneously maximize the expected returns and minimize the variance.

Jianshe (2005) developed a new theory of portfolio and risk based on incremental entropy and Markowitz's (1952) theory. He developed this theory by replacing arithmetic mean return adopted by Markowitz (1952), with geometric mean return as a criterion for assessing a portfolio. The new theory emphasizes that there is an objectively optimal portfolio for given probability of returns. Some portfolio optimization methodology has been discussed by Bugár and Uzsoki (2011) whereas other work related with diversification of investments has been provided by Markowitz (1959). Bera and Park (2008) remarked that Markowitz's (1952) mean-variance efficient portfolio selection is the one of the most widely used approaches in solving portfolio diversification problem. However, contrary to the notion of diversification, mean-variance approach often leads to portfolios highly concentrated on a few assets. In their paper, Bera and Park (2008) have proposed to use cross entropy measure as the objective function with side conditions coming from the mean and variance-covariance matrix of the resampled asset returns and illustrated their procedure with an application to the international equity indexes. Now since risk is associated with the concept of uncertainty, we should be able to develop measures of risk based on the concepts of divergence or cross entropy. We can develop such measures of divergence and then show how we can develop efficient frontiers for maximizing expected returns and simultaneously minimize measures of risk.

In the literature, there exist many well-known measures of divergence which find their applications to a variety of fields. One such measure is due to Kullback-Leibler (1951), which is an important measure of distance and is very useful in many real life situations. This measure is given by

$$D_{KL}(P;Q)=\sum_{i=1}^{n} p_i \log\frac{p_i}{q_i}\,. \qquad (1.1)$$

Recently, Parkash and Mukesh (2011) have introduced a new measure of cross entropy (divergence), given by

$$D(P;Q)=\sum_{i=1}^{n}\left(\frac{p_i^2}{q_i}+\frac{q_i^2}{p_i}-2p_i\right), \qquad (1.2)$$

and based on this divergence measure (1.2), Parkash and

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Mukesh (2012a) have developed an optimization principle for the measurement of risk in portfolio analysis.

It has been observed that generalized measures of cross entropy should be introduced because upon optimization, these measures lead to useful probability distributions and mathematical models in various disciplines. These generalized measures introduce flexibility in the system. Some parametric measures of directed divergence are:

$$D_R(P;Q)=\frac{1}{\alpha-1}\log\left(\sum_{i=1}^{n}p_i^{\alpha}q_i^{1-\alpha}\right),\ \alpha\neq 1, \alpha>0, \qquad (1.3)$$

which is Renyi's (1961) probabilistic measure of directed divergence.

$$D_{HC}(P;Q)=\frac{1}{\alpha-1}\left(\sum_{i=1}^{n}p_i^{\alpha}q_i^{1-\alpha}-1\right),\ \alpha\neq 1, \alpha>0, \qquad (1.4)$$

which is Havrada and Charvat's (1967) probabilistic measure of divergence. Some other interesting findings related with the literature of cross entropy have been provided by Taneja and Kumar (2004), Pardo (2003), Parkash and Mukesh (2012b) etc.

In the present communication, we introduce two new parametric measures of cross entropy and make their use for the measurement of risk in portfolio analysis. Before developing these measures, we need a brief introduction to the concept of mean-variance efficient frontier due to Markowitz (1952). This introduction has been provided by Kapur and Kesavan (1992) as explained below:

1.1 Markowitz (1952) mean-variance efficient frontier

Let π_j be the probability of the jth outcome for $j=1,2,\ldots,m$ and r_{ij} be the return on the ith security for $i=1,2,\ldots,n$ when the jth outcome occurs. Then the expected return on the ith security is given by

$$\bar{r}_i=\sum_{j=1}^{m}\pi_j r_{ij},\quad i=1,2,\ldots,n. \qquad (1.5)$$

Also, variances and covariances of returns are given by

$$\sigma_i^2=\sum_{j=1}^{m}\pi_j\left(r_{ij}-\bar{r}_i\right)^2,\ i=1,2,\ldots,n, \qquad (1.6)$$

and

$$\rho_{ik}\sigma_i\sigma_k=\sum_{j=1}^{m}\pi_j\left(r_{ij}-\bar{r}_i\right)\left(r_{kj}-\bar{r}_k\right),\ i,k=1,2,\ldots,n;\ i\neq k. \qquad (1.7)$$

Let a person decide to invest proportions $x_1,x_2,\ldots,x_n$ of his capital in n securities. Assume that $x_i\geq 0$ for all i, and that

$$\sum_{i=1}^{n}x_i=1. \qquad (1.8)$$

Then, the expected return and variance of the return are given by

$$E=\sum_{i=1}^{n}x_i\bar{r}_i, \qquad (1.9)$$

And

$$V=\sum_{i=1}^{n}x_i^2\sigma_i^2+2\sum_{k=1}^{n}\sum_{i<k}x_ix_k\rho_{ik}\sigma_i\sigma_k. \qquad (1.10)$$

Markowitz (1952) suggested that $x_1,x_2,\ldots,x_n$ be chosen to maximize E and to minimize V or alternatively, to minimize V when E is kept at a fixed value. Now

$$V=\sum_{j=1}^{m}\pi_j\left(x_1r_{1j}+x_2r_{2j}+\cdots+x_nr_{nj}-x_1\bar{r}_1-x_2\bar{r}_2-\cdots-x_n\bar{r}_n\right)^2$$

$$=\sum_{j=1}^{m}\pi_j\left(R_j-\bar{R}\right)^2, \qquad (1.11)$$

where

$$R_j=\sum_{i=1}^{n}x_ir_{ij}\quad\text{and}\quad \bar{R}=\sum_{i=1}^{n}x_i\bar{r}_i. \qquad (1.12)$$

that is, R_j is the return on investment when the jth outcome arises and $\bar{R}$ is the mean return on investment.

Next, we discuss an optimizational principle developed by Parkash and Mukesh (2012a) by using divergence measure (1.4).

1.2 Optimization Principle developed by Using Measure (1.2)

Markowitz's (1952) criterion for a choice from $x_1,x_2,\ldots,x_n$ was to minimize the variance, that is, to make $R_1,R_2,\ldots R_m$ as equal as possible among themselves. Any departure of $R_1,R_2,\ldots R_m$ from equality was considered a measure of risk. The same purpose can be accomplished if we choose $x_1,x_2,\ldots,x_n$ so as to minimize the directed divergence measure given by (1.2) of the distribution

$$P=\left(\frac{\pi_1R_1}{\sum_{j=1}^{m}\pi_jR_j},\frac{\pi_2R_2}{\sum_{j=1}^{m}\pi_jR_j},\cdots,\frac{\pi_mR_m}{\sum_{j=1}^{m}\pi_jR_j}\right)$$

from $\pi=(\pi_1,\pi_2,\ldots,\pi_m)$, that is, we choose $x_1,x_2,\ldots,x_n$ so as to minimize the following measure:

$$D(P;\pi)=\sum_{j=1}^{m}\left(\frac{P_j^2}{\pi_j}+\frac{\pi_j^2}{P_j}-2P_j\right)$$

$$=\sum_{j=1}^{m}\left(\frac{\pi_j^2R_j^2}{\bar{R}^2\pi_j}+\frac{\pi_j^2\bar{R}}{\pi_jR_j}-\frac{2\pi_jR_j}{\bar{R}}\right)$$

$$=\frac{1}{\bar{R}^2}\sum_{j=1}^{m}\pi_jR_j^2+\bar{R}\sum_{j=1}^{m}\frac{\pi_j}{R_j}-2, \qquad (1.13)$$

where $\sum_{j=1}^{m}\pi_jR_j=\sum_{j=1}^{m}\pi_j\sum_{i=1}^{n}x_ir_{ij}=\sum_{i=1}^{n}x_i\bar{r}_i=\bar{R}$. (1.14)

Thus, we can formulate an optimization principle as follows:

Choose $x_1,x_2,\ldots,x_n$ so as to minimize

$$\sum_{j=1}^{m}\pi_j\left(x_1r_{1j}+x_2r_{2j}+\cdots+x_nr_{nj}\right)^2+\sum_{j=1}^{m}\frac{\pi_j}{\left(x_1r_{1j}+x_2r_{2j}+\cdots+x_nr_{nj}\right)}, \qquad (1.15)$$

subject to

$$\sum_{j=1}^{m}\pi_j\left(x_1r_{1j}+x_2r_{2j}+\cdots+x_nr_{nj}\right)=\text{Constant}, \quad (1.16)$$

$$x_1+x_2+\cdots x_n=1, \quad (1.17)$$

and $x_1\geq 0, x_2\geq 0,\ldots,x_n\geq 0$.

Next, we propose two new parametric measures of cross entropy and study some of their essential properties.

2 New parametric measures of cross entropy

In this section, we consider the following set of all complete finite discrete probability distributions:

$$\Omega_n=\left\{P=(p_1,p_2,\ldots,p_n): p_i>0, \sum_{i=1}^{n}p_i=1\right\}, n\geq 2, \quad (2.1)$$

and introduce the following parametric measure of cross entropy.

2.1 One parametric measure of cross entropy

For $P,Q\in\Omega_n$, we propose a new parametric measure of cross entropy given by the following expression:

$$D_\alpha(P;Q)=\frac{\sum_{i=1}^{n}p_i\left(\alpha+\frac{1}{2}\right)^{\log\frac{p_i}{q_i}}-1}{\alpha-\frac{1}{2}}, \alpha>0, \ \alpha\neq\frac{1}{2}. \quad (2.2)$$

where α is a real parameter.

Note: We have

$$\lim_{\alpha\to\frac{1}{2}}D_\alpha(P;Q)=\lim_{\alpha\to\frac{1}{2}}\left[\frac{\sum_{i=1}^{n}p_i\left(\alpha+\frac{1}{2}\right)^{\log\frac{p_i}{q_i}}-1}{\alpha-\frac{1}{2}}\right]=\sum_{i=1}^{n}p_i\log\frac{p_i}{q_i},$$

which is Kullback-Leibler's (1951) measure of cross entropy. Thus, $D_\alpha(P;Q)$ is a generalized measure of cross entropy.

Some of the important properties of this cross entropy are:

1. $D_\alpha(P;Q)$ is a continuous function of $p_1,p_2,\ldots,p_n$ and $q_1,q_2,\ldots,q_n$.
2. $D_\alpha(P;Q)\geq 0$ and vanishes if and only if $P=Q$.
3. We can deduce from condition (2) that the minimum value of $D_\alpha(P;Q)$ is zero.
4. We shall now prove that $D_\alpha(P;Q)$ is a convex function of both P and Q. This result is important in establishing the property of global minimum.

Let

$$D_\alpha(P;Q)=f(p_1,p_2,\ldots,p_n;q_1,q_2,\ldots,q_n)$$

$$=\frac{\sum_{i=1}^{n}p_i\left(\alpha+\frac{1}{2}\right)^{\log\frac{p_i}{q_i}}-1}{\alpha-\frac{1}{2}}.$$

Thus

$$\frac{\partial f}{\partial p_i}=\frac{\left[1+\log\left(\alpha+\frac{1}{2}\right)\right]\left(\alpha+\frac{1}{2}\right)^{\log\frac{p_i}{q_i}}}{\alpha-\frac{1}{2}},$$

$$\frac{\partial^2 f}{\partial p_i^2}=\frac{\log\left(\alpha+\frac{1}{2}\right)\left[1+\log\left(\alpha+\frac{1}{2}\right)\right]\left(\alpha+\frac{1}{2}\right)^{\log\frac{p_i}{q_i}}}{\left(\alpha-\frac{1}{2}\right)p_i} \quad \forall\ i=1,2,\ldots,n,$$

and $\dfrac{\partial^2 f}{\partial p_i\partial p_j}=0\ \forall\ i,j=1,2,\ldots,n;\ i\neq j$.

Hence, the Hessian matrix of the second order partial derivatives of f with respect to $p_1,p_2,\ldots,p_n$ is given by $H_1=a_{ij}$, where

$$a_{ij}=\begin{cases}\dfrac{\log\left(\alpha+\frac{1}{2}\right)\left[1+\log\left(\alpha+\frac{1}{2}\right)\right]\left(\alpha+\frac{1}{2}\right)^{\log\frac{p_i}{q_i}}}{\left(\alpha-\frac{1}{2}\right)p_i}, & i=j,\\ 0, & i\neq j\end{cases}$$

which is positive definite. Similarly one can prove that the Hessian matrix of second order partial derivatives of f with respect to $q_1,q_2,\ldots,q_n$ is positive definite. Thus, we conclude that $D_\alpha(P;Q)$ is a convex function of both $p_1,p_2,\ldots,p_n$ and $q_1,q_2,\ldots,q_n$. Moreover, with the help of numerical data shown in the following table 1, we have presented $D_\alpha(P;Q)$ as shown in the following figure 1.

Table 1. $D_\alpha(P;Q)$ against p for $n=2,\alpha=10$.

p	q	$D_\alpha(P;Q)$
0.1	0.5	0.5906
0.2	0.5	0.3104
0.3	0.5	0.1310
0.4	0.5	0.0317
0.5	0.5	0.0000
0.6	0.5	0.0317
0.7	0.5	0.1310
0.8	0.5	0.3104
0.9	0.5	0.5906

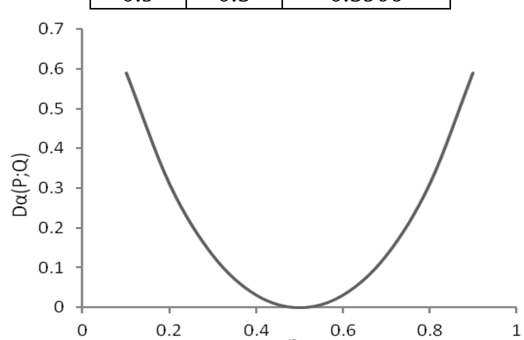

Fig. 1. Convexity of $D_\alpha(P;Q)$ with respect to P.

Under the above conditions, the function $D_\alpha(P;Q)$ is a valid parametric measure of cross entropy.

Next, we propose a two parametric measure of cross entropy.

2.2 Two parametric measure of cross entropy

For any $P,Q\in\Omega_n$, we propose a new parametric measure of cross entropy given by

$$D_{\alpha,\beta}(P;Q)=\frac{\sum_{i=1}^{n}p_i\left(\frac{\alpha}{2\beta}+\frac{1}{2}\right)^{\log\frac{p_i}{q_i}}-1}{\alpha-\beta},\ \alpha>0,\beta>0,\ \alpha\neq\beta. \quad (2.3)$$

Where α,β are real parameters representing some environmental factors, and the presence of these parameters gives a great deal of flexibility towards applications and also take into account the factors which might not have been possible otherwise.

Note: We have

$$\lim_{\substack{\alpha\to\frac{1}{2}\\ \beta\to\frac{1}{2}}}D_{\alpha,\beta}(P;Q)=\lim_{\substack{\alpha\to\frac{1}{2}\\ \beta\to\frac{1}{2}}}\left[\frac{\sum_{i=1}^{n}p_i\left(\frac{\alpha}{2\beta}+\frac{1}{2}\right)^{\log\frac{p_i}{q_i}}-1}{\alpha-\beta}\right]=\sum_{i=1}^{n}p_i\log\frac{p_i}{q_i}.$$

Thus, $D_{\alpha,\beta}(P;Q)$ is a generalization of Kullback-Leibler's (1951) measure of cross entropy.

Some of the important properties of this cross entropy are:

1. $D_{\alpha,\beta}(P;Q)$ is a continuous function of $p_1,p_2,\ldots,p_n$ and $q_1,q_2,\ldots,q_n$.
2. $D_{\alpha,\beta}(P;Q)\geq 0$ and vanishes if and only if $P=Q$.
3. We can deduce from condition (2) that the minimum value of $D_{\alpha,\beta}(P;Q)$ is zero.
4. We shall now prove that $D_{\alpha,\beta}(P;Q)$ is a convex function of both P and Q.

Let $D_{\alpha,\beta}(P;Q)=g(p_1,p_2,\ldots,p_n;q_1,q_2,\ldots,q_n)$

The Hessian matrix of the second order partial derivatives of g with respect to $p_1,p_2,\ldots,p_n$ is given by $H_2=b_{ij}$, where

$$b_{ij}=\begin{cases}\dfrac{\log\left(\frac{\alpha}{2\beta}+\frac{1}{2}\right)\left[1+\log\left(\frac{\alpha}{2\beta}+\frac{1}{2}\right)\right]\left(\frac{\alpha}{2\beta}+\frac{1}{2}\right)^{\log\frac{p_i}{q_i}}}{(\alpha-\beta)p_i}, & i=j\\ 0, & i\neq j\end{cases},$$

which is positive definite. A similar result is also true when we consider the partial derivatives of g with respect to $q_1,q_2,\ldots,q_n$. Thus, we conclude that $D_{\alpha,\beta}(P;Q)$ is a convex function of both $p_1,p_2,\ldots,p_n$ and $q_1,q_2,\ldots,q_n$. Moreover, we have presented $D_{\alpha,\beta}(P;Q)$ against p for $n=2,\alpha=10,\beta=15$ as shown in the following figure 2.

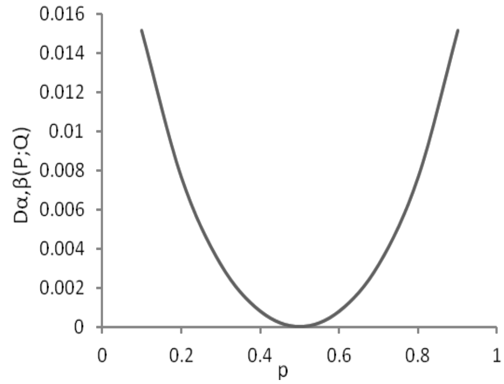

Fig. 2. Convexity of $D_{\alpha,\beta}(P;Q)$ with respect to P.

Under the above conditions, the function $D_{\alpha,\beta}(P;Q)$ is a valid parametric measure of cross entropy.

3 Measuring risk in portfolio analysis using parametric measures of cross entropy

In this section, we consider the following two cases for measuring risk in portfolio analysis by using two different parametric measures of cross entropy:

3.1 Measure of risk by using cross entropy (2.2)

Recently, Parkash and Mukesh (2012a) provided an optimization principle involving non-parametric measure of cross entropy (1.2) for the development of measures of risk when a person decides to invest proportions of his capitals in different securities. If we use one parametric measure of divergence developed in (2.2), we get a measure of risk in accordance with the optimization principle discussed in subsection 1.2. This measure is developed as follows:

$$R_1=\frac{\sum_{j=1}^{m}P_j\left(\alpha+\frac{1}{2}\right)^{\log\frac{P_j}{\pi_j}}-1}{\alpha-\frac{1}{2}}=\frac{\sum_{j=1}^{m}\frac{\pi_jR_j}{\bar{R}}\left(\alpha+\frac{1}{2}\right)^{\log\frac{\pi_jR_j}{\bar{R}\pi_j}}-1}{\alpha-\frac{1}{2}}$$

$$=\frac{\frac{1}{\bar{R}}\sum_{j=1}^{m}\pi_j\left[R_j\left(\alpha+\frac{1}{2}\right)^{\log\frac{R_j}{\bar{R}}}\right]-1}{\alpha-\frac{1}{2}}$$

$$=\frac{1}{\alpha-\frac{1}{2}}\left[\frac{1}{\bar{R}}E\left(R\left(\alpha+\frac{1}{2}\right)^{\log\frac{R}{\bar{R}}}\right)-1\right]. \quad (3.1)$$

If $\alpha<\frac{1}{2}$, minimizing the measure (3.1), we mean the maximization of expected utility of a person whose utility function is given by $u_1(x)=x\left(\alpha+\frac{1}{2}\right)^{\log\frac{x}{\bar{R}}}$. In this case the person is risk-averse. If $\alpha > \frac{1}{2}$, minimizing the measure (3.1), we mean minimization of the expected utility of a person whose utility function is given by $u_1(x) = x\left(\alpha + \frac{1}{2}\right)^{\log\frac{x}{\bar{R}}}$. In this case the person is risk-prone.

Thus, minimizing this measure implies the minimization of the expected utility of the risk-prone person and maximization of the expected utility of a risk-averse person.

3.2 Measure of risk by using cross entropy (2.3)

If we use the two parametric measure of cross entropy developed in (2.3), we can get another measure of risk discussed below:

$$R_2 = \frac{\sum_{j=1}^{m} P_j \left(\frac{\alpha}{2\beta} + \frac{1}{2}\right)^{\log\frac{P_j}{\pi_j}} - 1}{\alpha - \beta} = \frac{\sum_{j=1}^{m} \frac{\pi_j R_j}{\bar{R}} \left(\frac{\alpha}{2\beta} + \frac{1}{2}\right)^{\log\frac{\pi_j R_j}{\bar{R}\pi_j}} - 1}{\alpha - \beta}$$

$$= \frac{\frac{1}{\bar{R}} \sum_{j=1}^{m} \pi_j \left[R_j \left(\frac{\alpha}{2\beta} + \frac{1}{2}\right)^{\log\frac{R_j}{\bar{R}}} \right] - 1}{\alpha - \beta}$$

$$= \frac{1}{\alpha - \beta} \left[\frac{1}{\bar{R}} E\left(R \left(\frac{\alpha}{2\beta} + \frac{1}{2}\right)^{\log\frac{R}{\bar{R}}} \right) - 1 \right]. \qquad (3.2)$$

If $\alpha < \beta$, minimizing the measure (3.2), we mean the maximization of expected utility of a person whose utility function is given by $u_2(x) = x\left(\frac{\alpha}{2\beta} + \frac{1}{2}\right)^{\log\frac{x}{\bar{R}}}$. In this case the person is risk-averse. If $\alpha > \beta$, minimizing the measure (3.2), we mean minimization of the expected utility of a person whose utility function is given by $u_2(x) = x\left(\frac{\alpha}{2\beta} + \frac{1}{2}\right)^{\log\frac{x}{\bar{R}}}$. In this case the person is risk-prone.

Thus, minimizing this measure implies the minimization of the expected utility of the risk-prone person and maximization of the expected utility of a risk-averse person.

Concluding Remarks

Our study reveals that by using parametric measures of cross entropy, we can talk of maximizing the expected utility of risk-averse persons and of minimizing the expected utility of risk-prone persons. Such a study can be made available by the use of some other measures of cross entropy.

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